$A \xrightarrow{f} B$, $\quad$ $M = B$-mod.

$$\begin{tikzcd} B \arrow[r, "{(id, d)}"] & B \oplus M \\ A \arrow[u, "f"] \arrow[ur, "{(f,0)}"'] & \end{tikzcd}$$

$(b,m)\cdot(b',m') = (b\cdot b', bm'+b'm)$.

What's condition on d?
$$\begin{cases} d \text{ is additive} \\ d(f(a)) = 0. \\ d(b_1 b_2) = b_1\, db_2 + b_2\, db_1. \end{cases}$$

Defn: An A-der. of B into M is a d above.

Lemma / Defn: $A\text{-Der Mod} := \{(B \xrightarrow{d} M)\}$ has an initial obj
$$B \xrightarrow{d} \Omega_{B/A}$$

pf:
$$\Omega_{B/A} := \frac{\bigoplus_{b \in B} B \cdot db}{B\cdot\{da,\ d(b_1+b_2)-db_1-db_2,\ d(b_1b_2)-db_1-db_2\}}$$

$$\begin{tikzcd} db & \Omega_{B/A} \\ b \arrow[u, mapsto] & B \arrow[u] \end{tikzcd}$$

Lemma: ① $J \hookrightarrow C \twoheadrightarrow C/J$, then J/J^2 admits canonical C/J-mod structure

② $I := \ker(B \otimes_A B \xrightarrow{m} B)$

$\Omega_{B/A} \cong I/I^2$ (B-mod via ①)

$$\begin{tikzcd} 1\otimes b - b \otimes 1 & \Omega_{B/A} \\ b \arrow[u, mapsto] & B \arrow[u, "d"] \end{tikzcd}$$

pf of ②:
$$\begin{tikzcd} B \arrow[r, "{(id, d)}"] & B \oplus M \\ A \arrow[u, "f"] \arrow[ur, "{(f,0)}"'] & \end{tikzcd} \iff \begin{tikzcd} B \arrow[r, "{(id,d)}"] & B \oplus M \\ A \arrow[u, "f"] \arrow[r, "f"] & B \arrow[u, "{(id,0)}"'] \end{tikzcd} \iff \begin{tikzcd} & B & \\ B \otimes_A B / I^2 \arrow[ur, "m"] \arrow[rr] & & B \oplus M \arrow[ul, "pr_1"'] \\ & B \arrow[ul, "(id \otimes 1)"] \arrow[ur, "{(id,0)}"'] & \end{tikzcd}$$

$$\Updownarrow$$

B-linear $I/I^2 \to M$.

Lemma: $B = A[x_i; i\in I]/\underbrace{(f_\lambda(\underline{x}); \lambda\in\Lambda)}_{I}$, $\Omega_{B/A} \cong \mathrm{Coker}\left(I/I^2 \xrightarrow{\mathrm{Jac}} \bigoplus_{i\in I} B\cdot dx_i\right)$.

Example (3) in BB Notes

pf: univ. property of A-der.

Refn/Lemma: The cplx $\left[\underset{\deg -1}{I/I^2} \xrightarrow{\mathrm{Jac}} \underset{\deg 0}{\bigoplus_{i\in I} B\cdot dx_i}\right]$ is "up to homotopy" indep. of choice & functorial in f, denoted by $NL_{B/A}$ or NL_f.

idea: ① Say $I\subseteq A[\underline{x}] \twoheadrightarrow B \rightsquigarrow \left[I/I^2 \xrightarrow{\mathrm{Jac}} \bigoplus B\cdot dx_i\right]$

what if $J\subseteq A[\underline{x}][y] \twoheadrightarrow B \rightsquigarrow \left[J/J^2 \xrightarrow{\mathrm{Jac}} \bigoplus B\cdot dx_i \oplus B\,dy\right]$?

$$\begin{tikzcd}
{[I/I^2} \arrow[r, "\mathrm{Jac}"] \arrow[d] & {\bigoplus B\cdot dx_i]} \arrow[d] \\
{[J/J^2} \arrow[r, "\mathrm{Jac}"] \arrow[d] & {\bigoplus B\cdot dx_i \oplus B\,dy]} \arrow[d] \\
B\cdot\text{"}y-f(x)\text{"} \arrow[r, "\partial_y", "\cong"'] & B\cdot dy
\end{tikzcd}$$

⌜ $J = I + (y - f(x))$.

$\Rightarrow J/J^2 = I/I^2 \oplus B\cdot\text{"}y-f(x)\text{"}$ ⌟

(QED for independence).

②
$$\begin{tikzcd}
I \subseteq A[\underline{x}] \arrow[r, two heads] \arrow[d, dotted] \arrow[d, dotted, "\tilde g", shift left=6] & B \arrow[d, "g"] \\
J \subseteq A[\underline{y}] \arrow[r, two heads] & C
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
{[I/I^2} \arrow[r, "\mathrm{Jac}"] \arrow[d] & {\bigoplus B\cdot dx_i]} \arrow[d, "d\tilde g"] \\
{[J/J^2} \arrow[r, "\mathrm{Jac}"] & {\bigoplus C\cdot dy_j]}
\end{tikzcd}$$
B-cplx

diff. choice of $\tilde g \longleftrightarrow$ set theoretic $\{x_i\} \longrightarrow J$. $\rightsquigarrow \bigoplus B\cdot dx_i \xrightarrow{h} J/J^2$.

$\tilde g_1 \downarrow \downarrow \tilde g_2$ h $d\tilde g_1 \downarrow \downarrow d\tilde g_2$

homotopy.

$\square$

pf: $A \xrightarrow{f} B \xrightarrow{g} C \rightsquigarrow I\subseteq A[\underline{x}] \twoheadrightarrow B$

$J \subseteq B[\underline{y}] \twoheadrightarrow C$

$\tilde J \subseteq A[\underline{x}][\underline{y}] \twoheadrightarrow C$

$$\Rightarrow \begin{tikzcd}
 & I/I^2 \otimes_B C \arrow[r] \arrow[d, "\mathrm{Jac}\otimes_B C"] & \tilde J/\tilde J^2 \arrow[r] \arrow[d] & J/J^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus C\cdot dx_i \arrow[r] & \bigoplus C\cdot dx_i \oplus \bigoplus C\cdot dy_j \arrow[r] & \bigoplus C\cdot dy_j \arrow[r] & 0
\end{tikzcd}$$

w/ exact rows.

Lemma: $A \xrightarrow{f} B \xrightarrow{g} C$, then get natural exact seq.

$$(\star).\quad \pi_1(NL_{B/A} \overset{L}{\underset{B}{\otimes}} C) \to \pi_1(NL_{C/A}) \to \pi_1(NL_{C/B}) \to \Omega_{B/A} \underset{B}{\otimes} C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

In particular: • (First exact seq.) always:
$$\Omega_{B/A} \underset{B}{\otimes} C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

• (Second exact seq.) <u>If</u> $g: B \twoheadrightarrow C$ surj., then
$$\ker(g)/\ker(g)^2 \to \Omega_{B/A} \underset{B}{\otimes} C \to \Omega_{C/A} \to 0.$$
$$b \longmapsto db \otimes 1.$$

Property: (1) B/A f.g. $\Rightarrow$ $\Omega_{B/A}$ $/B$ f.g.

(2) $NL_{-/A}$ commutes w/ colim (in B).

(3) $NL_{B[1/b]/B} \overset{qis}{\sim} 0$; b/c $B[1/b] \cong B[x]/(bx-1)$.

$I/I^2 = B[1/b]\cdot(bx-1) =$ quotient of $B[1/b] \xrightarrow{Jac} B[1/b]\cdot dx$

$$bx-1 \longmapsto b\cdot dx.$$

So Jac surj $\Rightarrow$ $I/I^2 = B[1/b]\cdot(bx-1) = B[1/b]$ & Jac is an isom.

(2)+(3) $\Rightarrow$ (4) $NL_{S^{-1}B/B} \overset{qis}{\sim} 0. \Rightarrow S^{-1}B \underset{B}{\otimes} \Omega_{B/A} \xrightarrow{\simeq} \Omega_{S^{-1}B/A}$.

Exercise*: Show that $S^{-1}B \underset{B}{\otimes} \pi_1(NL_{B/A}) \cong \pi_1(NL_{S^{-1}B/A})$.

(4)
$$\begin{array}{ccc} B & \longrightarrow & B' := B \underset{A}{\otimes} A' \\ f\uparrow & & \uparrow f' \\ A & \longrightarrow & A' \end{array}$$

$$\begin{array}{ccccc} \rightsquigarrow B' \underset{B}{\otimes} NL_{B/A} & \simeq & [\, B' \underset{B}{\otimes} I/I^2 & \longrightarrow & \bigoplus B'\cdot dx_i \,] \\ \downarrow & & \downarrow & & \downarrow\cong \\ NL_{B'/A'} & \simeq & [\, I'/I'^2 & \longrightarrow & \bigoplus B'\cdot dx_i \,] \end{array}$$

$$\Rightarrow B' \underset{B}{\otimes} \Omega_{B/A} \cong \Omega_{B'/A'} \quad \& \quad \pi_1(B' \overset{L}{\underset{B}{\otimes}} NL_{B/A}) \twoheadrightarrow \pi_1(NL_{B'/A'}).$$

Exercise$^{☆}$: If $A \to A'$ flat, show $B' \otimes_B \pi_1(NL_{B/A}) \xrightarrow{\cong} NL_{B'/A'}$.

(5) (Exercise) Show $\Omega_{B_1 \otimes_A B_2 / A} \cong B \otimes_{B_1} \Omega_{B_1/A} \oplus B \otimes_{B_2} \Omega_{B_2/A}$.

Example (f.g. field extn): (1) L/k finite extn of fields.

$$k \hookrightarrow k_{sep} = F \hookrightarrow L.$$

Lemma: $NL_{F/k} \overset{qis}{\sim} 0$.

Step 1: Prove for $k(\alpha)/k$: $k(\alpha) = k[x]/P(x)$

$$\rightsquigarrow \quad k(\alpha) \cdot \text{"}P(x)\text{"} \xrightarrow{\text{Jac}} k(\alpha) \cdot dx$$

$$\text{"}P(x)\text{"} \longmapsto P'(x) \neq 0 \cdot dx.$$

Step 2: $F = k(\alpha_1, \cdots, \alpha_m) / k(\alpha_1, \cdots, \alpha_{m-1}) / \cdots / k$.

Use $(*)$ + induction $\square$.

$$\Rightarrow \quad NL_{L/k} \xrightarrow[\sim]{qis} NL_{L/F}.$$

Lemma: If L/F nontriv & purely insep. $NL_{L/F} \neq 0$, π_1 & $\Omega_{L/F}$ have same (fin.) dim as L-v.s.

pf: $L = \dfrac{F[x_1, \cdots, x_n]}{(x_1^p - a,\ x_2^p - f_1(x_1),\ x_3^p - f_2(x_1, x_2), \cdots, x_n^p - f_{n-1}(x_1, \cdots, x_{n-1}))}$

Check this is reg. seq. $\Rightarrow I/I^2 = L^{\oplus n}$

$$\Rightarrow \quad NL_{L/F} = \left[L^{\oplus n} \xrightarrow{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}} L^{\oplus n} \right] \quad \square.$$

(2) (Lemma) $k \to k(\underline{x})$ has $NL_{k(\underline{x})/k} \xrightarrow[\sim]{qis} k(\underline{x})^{\oplus \text{tr.deg}}$. (basis given by dx_i).

pf: $k \xrightarrow{f} k[\underline{x}] \xrightarrow{g} \mathrm{Frac}(k[\underline{x}]) = k(\underline{x})$.

$NL_f \simeq \oplus k[\underline{x}] \cdot dx_i$, $(*)$ + $NL_g \xrightarrow[\sim]{qis} 0$ gives rest.

Prop. K/k f.g. field extn. ① $\dim_K \Omega_{K/k} - \dim_K \pi_1(NL_{K/k}) = \text{tr.d.}(K/k)$.

② $\dim_K \Omega_{K/k} = \text{tr.d.}(K/k) \iff K$ is sep. gen. over k.

pf: ①: $k \longrightarrow k(\underline{x}) \xrightarrow{\text{fin. sep.}} k(\underline{x})_{sep} = F \xrightarrow{\text{purely insep.}} K$.

$$\Rightarrow \quad 0 \to \pi_1(NL_{K/k}) \to \pi_1(NL_{K/F}) \to \bigoplus_{i=1}^{tr.d.} K\cdot dx_i \to \Omega_{K/k} \to \Omega_{K/F} \to 0.$$

(2): Lemma: If $\cdot$ tr.d.$(K/k) > 0$, then $\exists$ tr. elt $x \in K$ s.t. $dx \neq 0$ in $\Omega_{K/k}$.
& $\cdot$ K/k f.g.

Warning: (1) Without K/k f.g., this is false.
(2) tr. elt $x \in K \not\Rightarrow dx \neq 0$ in $\Omega_{K/k}$.

pf: Choose elt $x \in K$ s.t. $dx \neq 0$ in $\Omega_{K/F}$.
If x transc., done, otherwise $x + x_1$.

Keep using Lemma $\Rightarrow$ get $k \subseteq k(x_1) \subseteq \cdots \subseteq k(x_1, \dots, x_n) \subseteq K$.
s.t. tr.d.$(K/k) = n$, $dx_i \neq 0$ in $\Omega_{K/k(x_1,\dots,x_{i-1})}$.

$$\Rightarrow \bigoplus_{i=1}^{n} K\cdot dx_i \xrightarrow{\cong} \Omega_{K/k}.$$

$$\Rightarrow K/k(x_1,\dots,x_n) \text{ fin. sep.} \qquad \square.$$

Lemma: k perf $\Rightarrow$ Every f.g. K/k is sep. gen.

pf idea:
$$\begin{tikzcd} K \arrow[r, "\text{fin. purely insep.}"] & K' \\ k \arrow[u] \arrow[ur, "\exists\ \text{sep.}"'] & \end{tikzcd}$$

same dim

same dim

$$\Rightarrow 0 \to \pi_1(NL_{K'/K}) \to K' \otimes_K \Omega_{K/k} \to \Omega_{K'/k} \to \Omega_{K'/K} \to 0$$

$$\Rightarrow \dim_K \Omega_{K/k} = \dim_{K'} \Omega_{K'/k} = \text{tr.d. } K'/k = \text{tr.d. } K/k. \qquad \square.$$

Prop. $k \to (B, m)$ local, $\to B/m = k'$.
Then (1) $\pi_1(NL_{k'/k}) \to m/m^2 \to \Omega_{B/k} \otimes_B k' \to \Omega_{k'/k} \to 0$
(2) If k'/k sep. gen. (= colim. of f. sep. g.),
then $\pi_1(NL_{k'/k}) = 0$.

Thm: k = field, B_0/k f.g., $B := (B_0)_{\mathfrak{p}}$, $k' := B/\mathfrak{p}$
$B_0 \supseteq \mathfrak{p}$

(1) $\Omega_{B/k}$ is free of rk $= \dim(B) + $ tr.d.(k'/k) $\Rightarrow$ B is reg.

(2) k is perfect, converse holds.

pf: (1) $\mathrm{tr.d.}(k'/k) + \dim(B) = \mathrm{rk}(\Omega_{B/k}) = \dim_{k'}(\Omega_{B/k} \otimes_B k')$

$\geq \dim_{k'}(m/m^2) + \dim_{k'} \Omega_{k'/k} - \dim_{k'} \pi_1(NL_{k'/k})$.

$= \dim_{k'}(m/m^2) + \mathrm{tr.d.}(k'/k)$.

$\Rightarrow \dim(B) \geq \dim_{k'}(m/m^2) \Rightarrow B$ is regular.

(2). k is perfect, any field extn is sep. gen.

$\Omega_{B/k}$ is a f.g. B-mod. $K := \mathrm{Frac}(B)$.

$\Omega_{B/k} \otimes_B K = \Omega_{K/k}$, $\dim_K \Omega_{K/k} = \mathrm{tr.d.}(K/k)$

$\Omega_{B/k} \otimes_B k'$ has $\dim_{k'} = \mathrm{tr.d.}(k'/k) + \underbrace{\dim_{k'}(m/m^2)}_{\subseteq \dim(B)}$.

Lemma: (B,m) = int'l local domain. M = f.g. B-mod.

M is fin. free $\iff \dim_{k'} M \otimes_B k' = \dim_K M \otimes_B K$ $(=n)$.

pf: Nakayama $\Rightarrow B^{\oplus n} \twoheadrightarrow M$

$- \otimes_B K$ is inj., hence inj. $\square$.

Let's globalize. Let $X \xrightarrow{f} Y$ morph. of schms.

Defn. • $\Omega^1_{X/Y} := \Delta_f^* \mathcal{I}_{\Delta_f}$. where $\Delta_f : X \longrightarrow X \times_Y X$.

$$\begin{array}{ccc} U = \mathrm{Spec}(B) & \subseteq & X \\ \downarrow & & \downarrow \\ V = \mathrm{Spec}(A) & \subseteq & Y \end{array}$$

then $\Omega^1_{X/Y}\big|_U \cong \widetilde{\Omega_{B/A}}$.

• $d: \mathcal{O}_X \longrightarrow \Omega^1_{X/Y}$, local section $b \in \mathcal{O}_X(U) \longmapsto \Delta_f^*(1\otimes b - b \otimes 1)$.

This is a map in $\mathrm{Shv}_{ab.gp.}(X)$, $f^{-1}\mathcal{O}_Y$-linear but NOT $\mathcal{O}_X$-linear.

- $\forall \mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$, $Y\text{-}\mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ defined similarly.

Then have universal property $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/Y}, \mathcal{F}) = Y\text{-}\mathrm{Der}(\mathcal{O}_X, \mathcal{F})$.

$$\alpha \longmapsto \alpha \circ d.$$

Properties : (1) $X \xrightarrow{f} Y$ loc. f.type $\Rightarrow \Omega^1_{X/Y}$ is locally f.g.

So if X Noeth. + f loc. f.t. $\Rightarrow \Omega^1_{X/Y} \in \mathrm{Coh}(X)$.

(2) Formation of $\Omega^1_{-/Y}$ commutes w/ localization

e.g. $(\Omega^1_{X/Y})_x \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$.

(3). commutes w/ base change in Y

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \qquad g'^*\Omega_f \cong \Omega_{f'}.$$

(4) $\searrow h$ $\qquad \Omega_h \cong f'^*\Omega_g \oplus g'^*\Omega_f$.

(5) $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow f^*\Omega_g \to \Omega_{g\circ f} \to \Omega_f \to 0$.

(6)
$$\begin{tikzcd} V(\mathcal{I}_Z) = Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$
imm.

$$i^*\mathcal{I}_Z \to i^*\Omega_f \to \Omega_g \to 0$$
$$i^*\mathcal{I}_Z = \mathcal{I}_Z/\mathcal{I}_Z^2$$

inj. if "$\pi_1(NL_{Z/Y}) = 0$".

defined globally : $\mathcal{I}_Z \subseteq \mathcal{O}_X \xrightarrow{d_{X/Y}} \Omega^1_{X/Y} \xrightarrow{1\otimes -} i^*\Omega_f$

$$\begin{tikzcd} \mathcal{I}_Z \arrow[d] & \\ \mathcal{I}_Z/\mathcal{I}_Z^2 \arrow[r, "\mathcal{O}_Z\text{-linear}"] & i^*\Omega_f \end{tikzcd}$$

Euler sequence : $X = \mathrm{Proj}(\underbrace{\mathbb{Z}[x_0, \dots, x_n]}_{= S}) = \mathbb{P}^n_{\mathbb{Z}}$

$\downarrow$

$Y = \mathrm{Spec}(\mathbb{Z})$

Heuristic: $\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1}\setminus\{0\} \xrightarrow{\pi} \mathbb{P}^n$ 1-forms on $\mathbb{P}^n$ pulls back to $\mathbb{G}_m$-equiv. 1-forms on $\mathbb{A}^{n+1}\setminus\{0\}$.

recording $\mathbb{G}_m$-weight

$$\Rightarrow \quad \begin{tikzcd} \pi^*\Omega^1_{\mathbb{P}^n} \arrow[r] & \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}}(-1)\cdot dX_i \arrow[r] \arrow[dr, "\text{Euler field. } \sum X_i \frac{\partial}{\partial X_i}"'] & \Omega^1_\pi \arrow[d, phantom, "\cong" rotate=90] \\ & & \mathcal{O}_{\mathbb{A}^{n+1}\setminus\{0\}} \end{tikzcd}$$

(b/c $\mathrm{Pic}(\mathbb{A}^{n+1}\setminus\{0\}) = \mathrm{Cl}(\mathbb{A}^{n+1}\setminus\{0\}) = \mathrm{Cl}(\mathbb{A}^{n+1}) = 0$).

pf: $M := \ker\left(\bigoplus_{i=0}^{n} S(-1)\cdot dX_i \xrightarrow{\sum X_i \frac{\partial}{\partial X_i}} S\right)$

$$(f_i \cdot dX_i) \longmapsto f_i \cdot X_i.$$

$\left\{ d\dfrac{X_i}{X_j} = \dfrac{X_j\, dX_i - X_i\, dX_j}{X_j^2} \right\}_{(i,j)\in\{0,\dots,n\}^2}$ defines $\Omega^1_{\mathbb{P}^n} \xrightarrow{\cong} \widetilde{M}$.

Prop. $0 \longrightarrow \Omega^1_{\mathbb{P}^n_{\mathbb{Z}}} \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(-1)\cdot \text{“}dX_i\text{”} \xrightarrow{\sum X_i \frac{\partial}{\partial X_i}} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$